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FREE VIBRATIONS OF FUNCTIONALLY GRADED PIEZOCERAMIC HOLLOW SPHERES WITH RADIAL POLARIZATION

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By introducing displacement functions as well as stress functions, two independent state equations with variable coefficients are established from the three-dimensional equations of a radially inhomogeneous spherically isotropic piezoelastic medium. By virtue of the laminated approximation method, the state equations are then transformed into the ones with constant variables in each layer, and the state variable solutions are presented. Based on the solutions, linear algebraic equations about the state variables only at the inner and outer spherical surfaces are derived by utilizing the continuity conditions at each interface. Frequency equations corresponding to two independent classes of vibrations are finally obtained from the free surface conditions. Numerical calculations are presented and the effect of the material gradient index on natural frequencies is discussed.

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1. INTRODUCTION

The mechanics of piezoelectric materials has been an important branch of solid mechanics in recent years. In particular, the research of piezoelectric plate and shell structures has attracted much attention from both engineers and scientists [1–3]. Great achievements have been made in the analysis of piezoelectric spherical shells. For example, Shu'lga [4] utilized separation formulae for displacements and stresses to simplify the basic equations of piezoelasticity for spherical isotropy, and obtained two independent classes of vibrations. Chen and Ding [5] exactly analyzed a rotating piezoelectric hollow sphere by introducing displacement functions. Chen *et al.* [6] recently investigated the coupled-free vibration problem of a submerged piezoelectric spherical shell.

Studies on functionally graded material (FGM) have been extensive in the last decade [7]. The dynamic analysis of FGM elastic plates and shells has been of particular research interest recently [8, 9]. Based on the three-dimensional elasticity equations for spherical isotropy, Chen *et al.* [10] exactly analyzed the coupled-free vibration of a fluid-filled FGM hollow sphere. By introducing displacement functions and using the Frobenius power-series method, Chen [11] recently considered the vibration problem of spherically isotropic piezoelastic spheres with a functionally graded property that the material constants vary with the radial co-ordinate in a power law. It should be noted that laminated models have

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been widely employed to analyze functionally graded materials or structures in the study of FGM [7, 12, 13]. However, with the increasing number of involved layers, conventional methods used by many authors usually lead to lower numerical efficiency because of the larger-scaled final solving matrix. The state-space method has shown to be very effective in the analysis of laminated structures because of the associated lower order solving matrix. Its recent applications in piezoelectric materials and structures can be found in references [14–17].

Based on the three-dimensional dynamic equations for spherically isotropic piezoelasticity, this study derives two decoupled state equations with variable coefficients through the introduction of displacement functions and stress functions. These two state equations can account for the material gradient characteristic along the radial direction. Since it is difficult to solve the state equations with variable coefficients directly, the laminated approximation method is employed for which the sphere is divided into a certain number of layers with a sufficiently small equal thickness. The state equations are then transformed approximately into the ones with constant coefficients within each layer. The matrix theory is then used to obtain the solutions, which give the relationships between the state variables at the upper surface and those at the lower surface of each layer. Allowing for the continuity conditions at each interface and the free conditions at the inner and outer boundary surfaces, frequency equations corresponding to two independent classes of vibrations are obtained. Numerical study is also given in the paper and the effect of the inhomogeneity parameter is discussed.

2. BASIC EQUATIONS

Toupin [18] has pointed out that a ceramic spherical shell before polarization is isotropic, but after the shell is permanently polarized in the radial direction by applying a large static voltage between its inner and outer surfaces, the point symmetry of the material is transversely isotropic with an axis of symmetry in the direction of the radius to the centre of the spherical shell. Such kind of transverse isotropy is also known as spherical isotropy. Ko and Pond [19] have used the constitutive relations for spherical isotropy to analyze a practical spherical multimode hydrophone. The basic equations of three-dimensional spherically isotropic piezoelasticity can be found in references [4, 11], for example. For the sake of the analysis to be followed, we give these equations here in a different way. Assuming the center of anisotropy coincident with the origin of spherical co-ordinate (r, θ, ϕ), the constitutive relations can be rewritten as follows:

$$
\begin{aligned}
\Sigma_{\theta\theta} &= r\sigma_{\theta\theta} = c_{11}S_{\theta\theta} + c_{12}S_{\phi\phi} + c_{13}S_{rr} + e_{31}\nabla_2\Phi,\\
\Sigma_{\phi\phi} &= r\sigma_{\phi\phi} = c_{12}S_{\theta\theta} + c_{11}S_{\phi\phi} + c_{13}S_{rr} + e_{31}\nabla_2\Phi,\\
\Sigma_{rr} &= r\sigma_{rr} = c_{13}S_{\theta\theta} + c_{13}S_{\phi\phi} + c_{33}S_{rr} + e_{33}\nabla_2\Phi,\\
\Sigma_{r\theta} &= r\sigma_{r\theta} = 2c_{44}S_{r\theta} + e_{15}\frac{\partial\Phi}{\partial\theta}, \quad \Sigma_{r\phi} = r\sigma_{r\phi} = 2c_{44}S_{r\phi} + \frac{e_{15}}{\sin\theta}\frac{\partial\Phi}{\partial\phi},\\
\Sigma_{\theta\phi} &= r\sigma_{\theta\phi} = 2c_{66}S_{\theta\phi},\\
\Delta_\theta &= rD_\theta = 2c_{15}S_{r\theta} - \varepsilon_{11}\frac{\partial\Phi}{\partial\theta}, \quad \Delta_\phi = rD_\phi = 2e_{15}S_{r\phi} - \frac{\varepsilon_{11}}{\sin\theta}\frac{\partial\Phi}{\partial\phi},\\
\Delta_r &= rD_r = e_{31}S_{\theta\theta} + e_{31}S_{\phi\phi} + e_{33}S_{rr} - \varepsilon_{33}\nabla_2\Phi, \qquad (1)
\end{aligned}
$$

where $\nabla_2 = r\partial/\partial r$, σ_{ij}, Φ and D_i are the stress tensor, electric potential and electric displacement vector, respectively, c_{ij}, ε_{ij} and e_{ij} are the elastic, dielectric and piezoelectric constants, respectively, and there is a relation $2c_{66} = c_{11} - c_{12}$. S_{ij} in equation (1) are determined by

$$
\begin{aligned}
& S_{rr} = rs_{rr} = \nabla_2 u_r,\ S_{\theta\theta} = rs_{\theta\theta} = \frac{\partial u_\theta}{\partial\theta} + u_r, \\
& S_{\phi\phi} = rs_{\phi\phi} = \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial\phi} + u_r + u_\theta\cot\theta, \\
& 2S_{r\theta} = 2rs_{r\theta} = \frac{\partial u_r}{\partial\theta} + \nabla_2 u_\theta - u_\theta, \qquad (2) \\
& 2S_{r\phi} = 2rs_{r\phi} = \frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} + \nabla_2 u_\phi - u_\phi, \\
& 2S_{\theta\phi} = 2rs_{\theta\phi} = \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{\partial u_\phi}{\partial\theta} - u_\phi\cot\theta,
\end{aligned}
$$

where s_{ij} is the strain tensor, u_i $(i = r, \theta, \phi)$ are three displacement components. The equations of motion can easily be transformed into the following form:

$$
\begin{aligned}
& \nabla_2\Sigma_{r\theta} + \csc\theta\frac{\partial\Sigma_{\theta\phi}}{\partial\phi} + \frac{\partial\Sigma_{\theta\theta}}{\partial\theta} + 2\Sigma_{r\theta} + (\Sigma_{\theta\theta} - \Sigma_{\phi\phi})\cot\theta = \rho r^2\frac{\partial^2 u_\theta}{\partial t^2}, \\
& \nabla_2\Sigma_{r\phi} + \csc\theta\frac{\partial\Sigma_{\phi\phi}}{\partial\phi} + \frac{\partial\Sigma_{\theta\phi}}{\partial\theta} + 2\Sigma_{r\phi} + 2\Sigma_{\theta\phi}\cot\theta = \rho r^2\frac{\partial^2 u_\phi}{\partial t^2}, \qquad (3) \\
& \nabla_2\Sigma_{rr} + \csc\theta\frac{\partial\Sigma_{r\phi}}{\partial\phi} + \frac{\partial\Sigma_{r\theta}}{\partial\theta} + \Sigma_{rr} - \Sigma_{\theta\theta} - \Sigma_{\phi\phi} + \Sigma_{r\theta}\cot\theta = \rho r^2\frac{\partial^2 u_r}{\partial t^2},
\end{aligned}
$$

where ρ is the density. The charge equation of electrostatics also can be rewritten as

$$
\nabla_2\Delta_r + \Delta_r + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\Delta_\theta\sin\theta) + \frac{1}{\sin\theta}\frac{\partial\Delta_\phi}{\partial\phi} = 0. \qquad (4)
$$

This paper shall consider the free vibrations of a functionally graded piezoceramic hollow sphere. We here assume that the elastic constants c_{ij}, the dielectric constants ε_{ij}, the piezoelectric constants e_{ij} as well as the mass density ρ all can be arbitrary functions of the radial co-ordinates r.

3. DERIVATION OF STATE EQUATIONS

The following separation formulae are employed:

$$
u_\theta = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{\partial G}{\partial\theta}, \quad u_\phi = \frac{\partial\psi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial G}{\partial\phi}, \quad u_r = w, \qquad (5)
$$

$$
\Sigma_{r\theta} = -\frac{1}{\sin\theta}\frac{\partial\Sigma_1}{\partial\phi} - \frac{\partial\Sigma_2}{\partial\theta}, \quad \Sigma_{r\phi} = \frac{\partial\Sigma_1}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\Sigma_2}{\partial\phi}, \qquad (6)
$$

where ψ, G and w are three displacement functions while Σ_1 and Σ_2 are two stress functions.

By employing equations (5) and (6), through some lengthy mathematical manipulations, we can transfer equations (1)–(4) into the following equations:

$$\nabla_2 \begin{Bmatrix} \Sigma_1 \\ \psi \end{Bmatrix} = \begin{bmatrix} -2 & -c_{66}(\nabla_1^2 + 2) + r^2\rho\partial^2/\partial t^2 \\ \dfrac{1}{c_{44}} & 1 \end{bmatrix} \begin{Bmatrix} \Sigma_1 \\ \psi \end{Bmatrix}, \tag{7}$$

$$\nabla_2 \begin{Bmatrix} \Sigma_{rr} \\ \Sigma_2 \\ G \\ w \\ \Delta_r \\ \Phi \end{Bmatrix} = \mathbf{P} \begin{Bmatrix} \Sigma_{rr} \\ \Sigma_2 \\ G \\ w \\ \Delta_r \\ \Phi \end{Bmatrix}, \tag{8}$$

where $\nabla_1^2 = \partial^2/\partial\theta^2 + \cot\theta(\partial/\partial\theta) + \csc^2\theta(\partial^2/\partial\phi^2)$ is the two-dimensional Laplacian operator on a spherical surface, $\mathbf{P}$ is an operator matrix, of which the non-zero elements are given as follows:

$$\begin{aligned}
&P_{11} = 2\beta - 1, \quad P_{12} = \nabla_1^2, \quad P_{13} = k_1\nabla_1^2, \\
&P_{14} = -2k_1 + r^2\rho\partial^2/\partial t^2, \quad P_{15} = 2P_{25} = -P_{64} = 2\gamma, \quad P_{21} = \beta, \\
&P_{22} = -2, \quad P_{23} = k_2\nabla_1^2 - 2c_{66} + r^2\rho\partial^2/\partial t^2, \quad P_{24} = -k_1, \\
&P_{32} = 1/c_{44}, \quad P_{33} = P_{34} = -P_{55} = 1, \quad P_{36} = e_{15}/c_{44}, \\
&P_{41} = \varepsilon_{33}/\alpha, \quad P_{43} = \beta\nabla_1^2, \quad P_{44} = -2\beta, \quad P_{45} = e_{33}/\alpha, \\
&P_{52} = e_{15}\nabla_1^2/c_{44}, \quad P_{56} = k_3\nabla_1^2, \\
&P_{61} = e_{33}/\alpha, \quad P_{63} = \gamma\nabla_1^2, \quad P_{65} = -c_{33}/\alpha,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
&\alpha = c_{33}\varepsilon_{33} + e_{33}^2, \quad \beta = (c_{13}\varepsilon_{33} + e_{31}e_{33})/\alpha, \\
&\gamma = (c_{13}e_{33} - c_{33}e_{31})/\alpha, \quad k_1 = 2(c_{13}\beta + e_{31}\gamma) - (c_{11} + c_{12}), \\
&k_2 = k_1/2 - c_{66}, \quad k_3 = \varepsilon_{11} + e_{15}^2/c_{44}.
\end{aligned} \tag{10}$$

For the non-axisymmetric free vibration of a closed spherical shell, it can be assumed that

$$\begin{aligned}
&\Sigma_1 = bc_{44}^0 \sum_{m=0}^{n}\sum_{n=0}^{\infty} \Sigma_{1n}(\xi)S_n^m(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \psi = b\sum_{m=0}^{n}\sum_{n=0}^{\infty} \psi_n(\xi)S_n^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t}, \\
&\Sigma_{rr} = bc_{44}^0 \sum_{m=0}^{n}\sum_{n=0}^{\infty} \Sigma_{rn}(\xi)S_n^m(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \Sigma_2 = bc_{44}^0 \sum_{m=0}^{n}\sum_{n=0}^{\infty} \Sigma_{2n}(\xi)S_n^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t}, \\
&G = b\sum_{m=0}^{n}\sum_{n=0}^{\infty} G_n(\xi)S_n^m(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad w = b\sum_{m=0}^{n}\sum_{n=0}^{\infty} w_n(\xi)S_n^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t}, \\
&\Delta_r = be_{33}^0 \sum_{m=0}^{n}\sum_{n=0}^{\infty} \Delta_{rn}(\xi)S_n^m(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad \Phi = (be_{33}^0/\varepsilon_{33}^0)\sum_{m=0}^{n}\sum_{n=0}^{\infty} \Phi_n(\xi)S_n^m(\theta,\phi)\mathrm{e}^{\mathrm{i}\omega t},
\end{aligned} \tag{11}$$

where $S_n^m(\theta, \phi) = P_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$ are the spherical harmonic functions and $P_n^m(x)$ are the associated Legendre polynomials, n and m are integers, ω is the circular frequency, $\xi = r/b$ is the dimensionless radial co-ordinate, b is the outer radius of the spherical shell, c_{44}^0 represents the value of c_{44} at the inner surface $r = a$, i.e., $c_{44}^0 = c_{44}|_{r=a}$, and so on. Substituting equation (11) into equations (7) and (8), gives

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{V}_k(\xi) = \mathbf{M}_k(\xi)\mathbf{V}_k(\xi) \quad (k = 1, 2), \tag{12}$$

where $\mathbf{V}_1 = [\Sigma_{1n}, \psi_n]^{\mathrm{T}}$, $\mathbf{V}_2 = [\Sigma_{rn}, \Sigma_{2n}, G_n, w_n, \Delta_{rn}, \Phi_n]^{\mathrm{T}}$, and

$$\mathbf{M}_1 = \frac{1}{\xi}\begin{bmatrix} -2 & \dfrac{(l-2)c_{66}}{c_{44}^0} - J \\ \dfrac{c_{44}^0}{c_{44}} & 1 \end{bmatrix}, \tag{13}$$

$$\mathbf{M}_2 = \frac{1}{\xi}\begin{bmatrix} 2\beta - 1 & -l & -\dfrac{k_1 l}{c_{44}^0} & -\dfrac{2k_1}{c_{44}^0} - J & \dfrac{2\gamma e_{33}^0}{c_{44}^0} & 0 \\ \beta & -2 & -\dfrac{k_2 l + 2c_{66}}{c_{44}^0} - J & -\dfrac{k_1}{c_{44}^0} & \dfrac{\gamma e_{33}^0}{c_{44}^0} & 0 \\ 0 & \dfrac{c_{44}^0}{c_{44}} & 1 & 1 & 0 & \dfrac{e_{15}e_{33}^0}{c_{44}\varepsilon_{33}^0} \\ \dfrac{c_{44}^0\varepsilon_{33}}{\alpha} & 0 & -\beta l & -2\beta & \dfrac{e_{33}e_{33}^0}{\alpha} & 0 \\ 0 & -\dfrac{e_{15}c_{44}^0 l}{e_{33}^0 c_{44}} & 0 & 0 & -1 & -\dfrac{k_3 l}{\varepsilon_{33}^0} \\ \dfrac{e_{33}c_{44}^0\varepsilon_{33}^0}{\alpha e_{33}^0} & 0 & -\dfrac{\gamma\varepsilon_{33}^0 l}{e_{33}^0} & -\dfrac{2\gamma\varepsilon_{33}^0}{e_{33}^0} & -\dfrac{c_{33}\varepsilon_{33}^0}{\alpha} & 0 \end{bmatrix}, \tag{14}$$

where $l = n(n+1)$, $J = \xi^2\Omega^2\rho/\rho^0$, $\Omega^2 = b^2\omega^2\rho^0/c_{44}^0$ is the non-dimensional frequency, $\rho^0 = \rho|_{r=a}$ is the value of the mass density at the inner surface. At this stage, we have established two separated state equations with variable coefficients directly based on the three-dimensional piezoelasticity equations.

4. LAMINATED APPROXIMATION THEORY AND THE SOLUTIONS

Since it is difficult to solve the state equations with variable coefficients directly, we here intend to employ the laminated approximation theory [12, 13] to turn them into the ones with constant coefficients. We first divide the sphere into p equal layers (see Figure 1), each with a sufficiently small thickness h/p, here $h = b - a$ is the thickness of the shell. Since each layer is thin enough, the material constants within it can be assumed constant rather than variable. In the following, their values at each mediate plane are to be taken, i.e., in equations (13) and (14), we have $c_{44} = c_{44}|_{r=a+(2j-1)h/2p}$, etc. in the jth layer. Meanwhile, since the matrix $\mathbf{M}_k$ still includes the variable ξ, we make it equal its value also at the mediate plane, i.e., $\xi = \xi_0 + (2j-1)(1-\xi_0)/(2p)$ in the jth layer, here $\xi_0 = a/b$ is the ratio

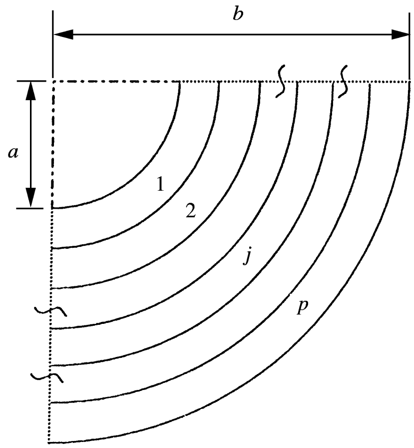

Figure 1. The geometry of a hollow sphere divided into p layers.

between the inner radius and the outer radius. Thus, the coefficient matrix $\mathbf{M}_k$ in equation (12) becomes constant within the jth layer (denoted as $\mathbf{M}_k^j$ in the following) and the solutions can be obtained as follows:

$$\mathbf{V}_k(\xi) = \exp[(\xi - \xi_{j-1})\mathbf{M}_k^j]\mathbf{V}_k(\xi_{j-1}) \quad (\xi_{j-1} \leqslant \xi \leqslant \xi_j, \quad j = 1, 2, \ldots, p), \tag{15}$$

where $\xi_j = \xi_0 + j(1 - \xi_0)/p$. The exponential matrix $\exp[(\xi - \xi_{j-1})\mathbf{M}_k^j]$ is known as the transfer matrix, which can be expressed in terms of a polynomial about the matrix $\mathbf{M}_k^j$ by virtue of the Cayley–Hamilton theorem [20].

The continuity conditions at each interface demand that the eight state variables be continuous. Thus one obtains from equation (15)

$$\mathbf{V}_k(1) = \mathbf{T}_{kn}\mathbf{V}_k(\xi_0) \quad (n = 1, 2, 3, \ldots), \tag{16}$$

where $\mathbf{T}_{1n} = \Pi_{j=p}^{1}\exp[(1 - \xi_0)\mathbf{M}_1^j/p]$ and $\mathbf{T}_{2n} = \Pi_{j=p}^{1}\exp[(1 - \xi_0)\mathbf{M}_2^j/p]$ are matrices of the second and sixth order respectively.

It can be seen from equations (5) and (6) that Σ_{10}, ψ_0, Σ_{20} and G_0 contribute nothing to the electroelastic field and hence can be assumed to be zero. In fact, equation (12) degenerates to the following equation when $n = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{Bmatrix} \Sigma_{r0} \\ w_0 \\ \Delta_{r0} \\ \Phi_0 \end{Bmatrix} = \frac{1}{\xi}\begin{bmatrix} 2\beta - 1 & -\dfrac{2k_1}{c_{44}^0} - J & \dfrac{2\gamma e_{33}^0}{c_{44}^0} & 0 \\ \dfrac{\varepsilon_{33}c_{44}^0}{\alpha} & -2\beta & \dfrac{e_{33}e_{33}^0}{\alpha} & 0 \\ 0 & 0 & -1 & 0 \\ \dfrac{e_{33}c_{44}^0\varepsilon_{33}^0}{\alpha e_{33}^0} & -\dfrac{2\gamma\varepsilon_{33}^0}{e_{33}^0} & -\dfrac{c_{33}\varepsilon_{33}^0}{\alpha} & 0 \end{bmatrix}\begin{Bmatrix} \Sigma_{r0} \\ w_0 \\ \Delta_{r0} \\ \Phi_0 \end{Bmatrix}. \tag{17}$$

We can obtain similarly

$$\begin{Bmatrix} \Sigma_{r0}(1) \\ w_0(1) \\ \Delta_{r0}(1) \\ \Phi_0(1) \end{Bmatrix} = \mathbf{T}_{20} \begin{Bmatrix} \Sigma_{r0}(\xi_0) \\ w_0(\xi_0) \\ \Delta_{r0}(\xi_0) \\ \Phi_0(\xi_0) \end{Bmatrix} \quad (n = 0), \tag{18}$$

where $\mathbf{T}_{20}$ is a fourth order square matrix.

5. FREQUENCY EQUATIONS

For the free vibration problem, we have the following boundary conditions at the inner and outer spherical surfaces:

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = D_r = 0 \quad (r = a, b). \tag{19}$$

Equation (19) can be expressed in terms of the state variables as follows:

$$\Sigma_{rr} = \Sigma_1 = \Sigma_2 = \Delta_r = 0 \quad (\xi = \xi_0, 1). \tag{20}$$

It can be seen that either the controlling equations as well as the boundary and continuity conditions all can be separated into two independent classes: The first one is only related to two state variables Σ_1 and ψ, while the second one is related to the other six state variables. Utilizing equation (20), one gets from equations (16) and (18)

$$\begin{Bmatrix} 0 \\ \psi_n(1) \end{Bmatrix} = \mathbf{T}_{1n} \begin{Bmatrix} 0 \\ \psi_n(\xi_0) \end{Bmatrix} \quad (n = 1, 2, 3, \dots), \tag{21}$$

$$\begin{Bmatrix} 0 \\ 0 \\ G_n(1) \\ w_n(1) \\ 0 \\ \Phi_n(1) \end{Bmatrix} = \mathbf{T}_{2n} \begin{Bmatrix} 0 \\ 0 \\ G_n(\xi_0) \\ w_n(\xi_0) \\ 0 \\ \Phi_n(\xi_0) \end{Bmatrix} \quad (n = 1, 2, 3, \dots), \tag{22}$$

$$\begin{Bmatrix} 0 \\ w_0(1) \\ 0 \\ \Phi_0(1) \end{Bmatrix} = \mathbf{T}_{20} \begin{Bmatrix} 0 \\ w_0(\xi_0) \\ 0 \\ \Phi_0(\xi_0) \end{Bmatrix} \quad (n = 0). \tag{23}$$

For non-trivial solutions, equations (21)–(23) give the frequency equations of two independent classes of vibrations, respectively,

$$T_{1n12} = 0 \quad (n = 1, 2, 3, \dots) \tag{24}$$

for the first class, and

$$T_{2012} = 0 \quad (n = 0), \tag{25}$$

$$\begin{vmatrix} T_{2n13} & T_{2n14} & T_{2n16} \\ T_{2n23} & T_{2n24} & T_{2n26} \\ T_{2n53} & T_{2n54} & T_{2n56} \end{vmatrix} = 0 \quad (n = 1, 2, 3, \ldots) \tag{26}$$

for the second class. In equations (24)–(26), T_{knij} represents the element on the ith row and jth column of the matrix $\mathbf{T}_{kn}$. It should be noted that the frequency equation of the second class for $n = 0$ is shown in equation (25) rather than the following equation:

$$\begin{vmatrix} T_{2012} & T_{2014} \\ T_{2032} & T_{2034} \end{vmatrix} = 0. \tag{27}$$

This follows because, when the inner spherical surface is free from the normal electric displacement the outer surface will naturally be free from the normal electric displacement as one can see from equation (17) directly. Thus, the third equation in equation (23) is automatically satisfied while the first equation gives rise to equation (25).

It is noted here that since the first class of vibration is only related to Σ_1 and ψ and no electric parameters are involved, it is exactly the same as that for the corresponding elastic sphere.

For calculating the mode shapes, once the frequency is obtained, the eigenstate vectors at the inner and/or outer spherical surfaces can be solved from equations (21)–(23). The state vectors at any interior point can then be calculated by the following formula:

$$\mathbf{V}_k(\xi) = \exp[(\xi - \xi_{j-1})\mathbf{M}_k^j] \prod_{i=j-1}^{1} \exp[(1 - \xi_0)\mathbf{M}_k^i/p]\mathbf{V}_k(\xi_0) \quad (\xi_{j-1} \leqslant \xi \leqslant \xi_j). \tag{28}$$

The induced variables are determined by

$$\begin{aligned}
\Sigma_{\theta\theta} - \Sigma_{\phi\phi} &= 2c_{66}\left(\nabla_1^2 G - 2\frac{\partial^2 G}{\partial\theta^2} + 2\cot\theta\csc\theta\frac{\partial\psi}{\partial\phi} - 2\csc\theta\frac{\partial^2\psi}{\partial\theta\partial\phi}\right), \\
\Sigma_{\theta\theta} + \Sigma_{\phi\phi} &= 2\beta\Sigma_{rr} + 2\gamma\Delta_r + k_1\nabla_1^2 G - 2k_1 w, \\
\Sigma_{0\phi} &= -c_{66}\left(\nabla_1^2\psi - 2\frac{\partial^2\psi}{\partial\theta^2} + 2\cot\theta\csc\theta\frac{\partial G}{\partial\phi} + 2\csc\theta\frac{\partial^2 G}{\partial\theta\partial\phi}\right), \\
\Delta_\theta &= -\frac{e_{15}}{c_{44}}\left(\frac{1}{\sin\theta}\frac{\partial\Sigma_1}{\partial\phi} + \frac{\partial\Sigma_2}{\partial\theta}\right) - k_3\frac{\partial\Phi}{\partial\theta}, \\
\Delta_\phi &= \frac{e_{15}}{c_{44}}\left(\frac{\partial\Sigma_1}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\Sigma_2}{\partial\phi}\right) - k_3\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}.
\end{aligned} \tag{29}$$

6. NUMERICAL EXAMPLES

It is pointed out that there are an infinite number of frequencies for each class of vibration due to the three-dimensional property of the resulting frequency equations. In the following, we only give out the lowest non-zero natural frequencies ($\Omega > 0$) that are of the most

TABLE 1

Material constants of PZT-4 and ZnO
(*Units*: c_{ij}—10^{10} N/m^2, e_{ij}—C/m^2, ε_{ij}—10^{-11} F/m, ρ—kg/m^3)

Materials	Material constants
PZT-4	$c_{11} = 13{\cdot}9$, $c_{12} = 7{\cdot}8$, $c_{13} = 7{\cdot}4$, $c_{33} = 11{\cdot}5$, $c_{44} = 2{\cdot}56$, $e_{15} = 12{\cdot}7$, $e_{31} = -5{\cdot}2$, $e_{33} = 15{\cdot}1$, $\varepsilon_{11} = 650$, $\varepsilon_{33} = 560$, $\rho = 7500$
ZnO	$c_{11} = 20{\cdot}97$, $c_{12} = 12{\cdot}11$, $c_{13} = 10{\cdot}51$, $c_{33} = 21{\cdot}09$, $c_{44} = 4{\cdot}25$, $e_{15} = -0{\cdot}59$, $e_{31} = -0{\cdot}61$, $e_{33} = 1{\cdot}14$, $\varepsilon_{11} = 7{\cdot}38$, $\varepsilon_{33} = 7{\cdot}83$, $\rho = 5676$

TABLE 2

Comparison and convergence study of the present method

		The first class				The second class			
n		1	2	3	4	0	1	2	3
$\lambda = 0$	$p = 19$	5·96091	2·71474	4·19810	5·53388	6·05719	4·84009	2·82733	4·65537
	$p = 20$	5·96094	2·71459	4·19783	5·53351	6·05722	4·84006	2·82735	4·65530
	Chen [11]	5·96125	2·71320	4·19541	5·53001	6·05750	4·83984	2·82725	4·65467
$\lambda = 10$	$p = 19$	8·65683	3·76146	5·83179	7·69518	8·20587	6·35983	3·57541	5·76395
	$p = 20$	8·65693	3·76126	5·83144	7·69467	8·20575	6·35965	3·57531	5·76379

importance in practical engineering. Notice that when $n = 1$, there are modes corresponding to the rigid-body rotation and rigid-body translation for the first and second classes of vibrations, respectively, and when $n = 0$, there is a mode of constant potential. In these cases, the natural frequency equals zero.

Consider an FGM piezoceramic hollow sphere with the inner-radius-to-outer-radius ratio $\xi_0 = 0{\cdot}3$. The functionally graded property is represented by [21]

$$M = M_P\left(\frac{h-z}{h}\right)^{\lambda} + M_z\left[1 - \left(\frac{h-z}{h}\right)^{\lambda}\right], \tag{30}$$

where $z = r - a$, λ is the inhomogeneity parameter or gradient index, M indicates an arbitrary material constant, and M_P and M_z are the material constants of PZT-4 and ZnO respectively [22, 23]. It is noted here that the three material constants c_{13}, c_{33} and c_{44} of PZT-4 given in reference [23] are wrong when they are cited from Dieulesaint and Royer [22]. These material constants are listed in Table 1 for the readers' convenience.

The first attempt here is to verify the validity as well as the covergence characteristic of the method. Table 2 gives the lowest natural frequencies (Ω) of the first and second classes, respectively, calculated when the sphere is divided into 19 equal layers and 20 equal layers. Two values of gradient index are adopted. In particular, for a homogeneous sphere ($\lambda = 0$), results are compared to that obtained using different three-dimensional methods [11]. It is seen that good agreement is obtained between the two methods, which validates the present method. Furthermore, the difference between the results of $p = 19$ and 20 is completely

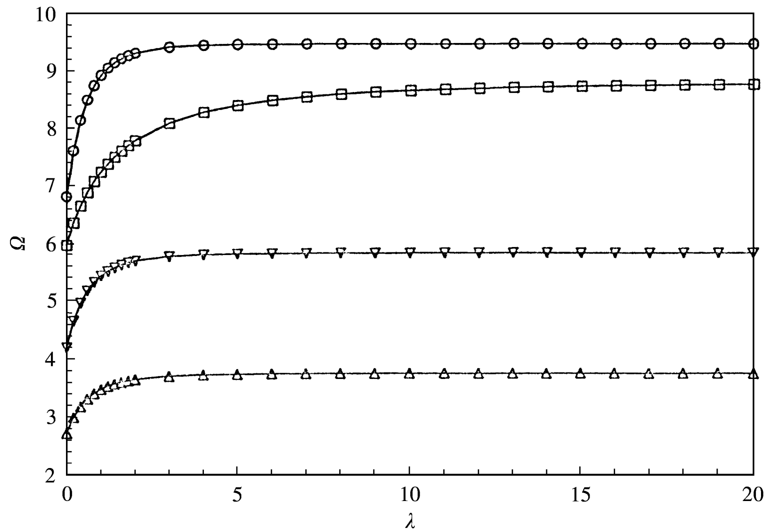

Figure 2. Variation of Ω versus λ for the first class: —□—, $n = 1$; —△—, $n = 2$; —▽—, $n = 3$; —○—, $n = 5$.

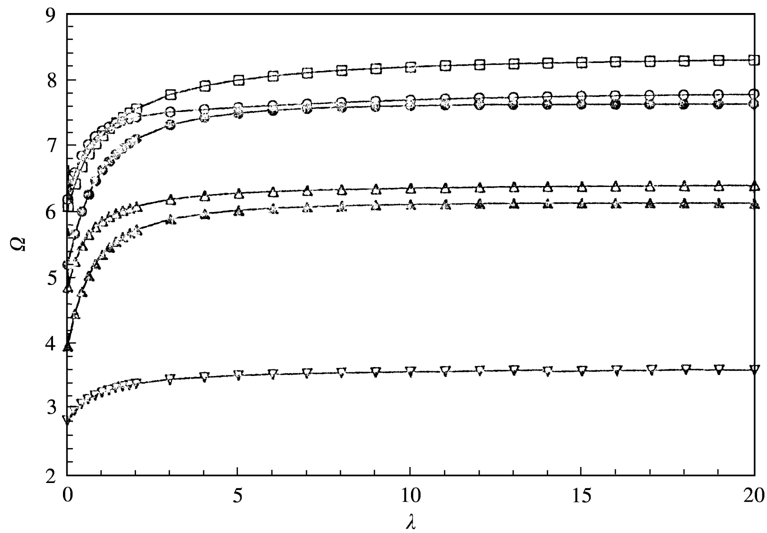

Figure 3. Variation of Ω versus λ for the second class: —□—, $n = 0$; —△—, $n = 1$; —▽—, $n = 2$; —○—, $n = 4$; —▲—, $n = 1$ (elastic); —■—, $n = 4$ (elastic).

negligible. Thus in the following, we take $p = 20$ and the results are believed to be with high precision.

Figures 2 and 3 display curves of the lowest non-dimensional frequency Ω of the first and second classes, respectively, versus the gradient index λ. From the figures, it is shown that with the increase of the gradient index, the natural frequency increases for all modes considered. The variation is more significant when $0 \leqslant \lambda \leqslant 5$. It should be noted that the frequencies of the sphere for $\lambda = 0$ correspond to those of a homogeneous PZT-4 hollow sphere, while those for $\lambda \to \infty$ corresponds to a homogeneous ZnO hollow sphere. This is obvious as we can see from equation (30). To show the piezoelectric effect, two curves corresponding to the FGM elastic sphere neglecting the piezoelectric effect are simultaneously given in Figure 3 for two mode numbers $n = 1$ and 4. It is seen that for an FGM sphere, the piezoelectric effect raises the natural frequency and the increment is also associated with the gradient index.

7. CONCLUDING REMARKS

(1) The two state equations presented here are order reduced when compared to the original controlling equations that will be convenient for solving practical problems. For the free vibration problem, it is concluded rather naturally that there are two independent classes of vibrations. In particular, there is no piezoelectric or dielectric parameter involved in the first class, which is actually identical to the one of the corresponding spherically isotropic elastic sphere.
(2) It also can be seen that the integer m, which represents the non-axisymmetric characteristic of the vibration, does not appear in the frequency equations. This is because any non-axisymmetric modes of vibrations can be obtained by the superposition of the axisymmetric ones with respect to different oriented spherical co-ordinates of identical natural frequency. The explanation has been given in detail by Silbiger [24] in the case of a homogeneous, isotropic, elastic hollow sphere.
(3) The present method is completely based upon three-dimensional piezoelasticity without introducing any assumptions on deformation. The only approximation is obtained when the laminated model is employed. However, as the number of layers increases, the present solution will gradually approach the exact solution. Therefore, the present method may be a benchmark for verifying two-dimensional plate/shell theories or numerical methods. It also can be employed to analyze the free vibrations of multilayered piezoceramic hollow spheres.
(4) Numerical results show that the effect of the material gradient index on natural frequencies is significant, especially in the interval $\lambda \in [0, 5]$. This point may be very important for the practical design of FGM plates and shells.

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REFERENCES

1. P. BISEGNA and F. MACERI 1996 *Journal of Applied Mechanics* **63**, 628–638. An exact three-dimensional solution for simply supported rectangular piezoelectric plates.
2. S. KAPURIA, P. C. DUMIR and S. SENGUPTA 1996 *Computers and Structures* **61**, 1085–1099. Exact piezothermoelastic axisymmetric solution of a finite transversely isotropic cylindrical shell.
3. H. J. DING, W. Q. CHEN, Y. M. GUO and Q. D. YANG 1997 *International Journal of Solids and Structures* **34**, 2025–2034. Free vibrations of piezoelectric cyindrical shells filled with compressible fluid.
4. N. A. SHUL'GA 1993 *Soviet Applied Mechanics* **29**, 812–817. Harmonic electroelastic oscillation of spherical bodies.
5. W. Q. CHEN and H. J. DING 1998 *Acta Mechanica Sinica* **14**, 257–265. Exact static analysis of a rotating piezoelectric spherical shell.
6. W. Q. CHEN, H. J. DING and R. Q. XU 2001 *Computers and Structures* **79**, 653–663. Three dimensional free vibration analysis of a fluid-filled piezoceramic hollow sphere.

7. Y. TANIGAWA 1995 *Applied Mechanics Reviews* **48**, 287–300. Some basic thermoelastic problems for nonhomogeneous structural materials.
8. C. T. LOY, K. Y. LAM and J. N. REDDY 1999 *International Journal of Mechanical Sciences* **41**, 309–324. Vibration of functionally graded cylindrical shells.
9. Z. Q. CHENG and R. C. BATRA 2000 *Journal of Sound and Vibration* **229**, 879–895. Exact correspondence between eigenvalues of membranes and functionally graded simply supported polygonal plates.
10. W. Q. CHEN, X. WANG and H. J. DING 1999 *Journal of the Acoustical Society of America* **106**, 2588–2594. Free vibration of a fluid-filled hollow sphere of a functionally graded material with spherical isotropy.
11. W. Q. CHEN 2000 *Journal of Sound and Vibration* **229**, 833–860. Vibration theory of non-homogeneous, spherically isotropic piezoelastic bodies.
12. Y. OOTAO and Y. TANIGAWA 2000 *International Journal of Solids and Structures* **37**, 4377–4401. Three-dimensional transient piezothermoelasticity in functionally graded rectangular plate bonded to a piezoelectric plate.
13. B. L. WANG, J. C. HAN and S. Y. DU 1999 *Theoretical and Applied Fracture Mechanics* **32**, 165–175. Functionally graded penny-shaped cracks under dynamic loading.
14. W. Q. CHEN, J. LIANG and H. J. DING 1997 *Acta Materiae Compositae Sinica* **14**, 108–115. Three dimensional analysis of bending problems of thick piezoelectric composite rectangular plates (in Chinese).
15. W. Q. CHEN, R. Q. XU and H. J. DING 1998 *Journal of Sound and Vibration* **218**, 741–748. On free vibration of a piezoelectric composite rectangular plate.
16. H. J. DING, R. Q. XU and F. L. GUO 1999 *Science in China* (*E*) **42**, 388–395. Exact axisymmetric solution of laminated transversely isotropic piezoelectric circular plates (I)—exact solutions for piezoelectric circular plate.
17. J. G. WANG 1999 *Science in China* (*A*) **42**, 1323–1331. State vector solutions for nonaxisymmetric problem of multilayered half space piezoelectric medium.
18. R. A. TOUPIN 1959 *Journal of the Acoustical Society of America* **31**, 315–318. Piezoelectric relations and the radial deformation of a polarized spherical shell.
19. S. H. KO and H. L. POND 1978 *Journal of the Acoustical Society of America* **64**, 1270–1277. Improved design of spherical multimode hydrophone.
20. R. BELLMAN 1970 *Introduction of Matrix Analysis*. New York: McGraw-Hill.
21. J. N. REDDY, C. M. WANG and S. KITIPORNCHAI 1999 *European Journal of Mechanics, A/Solids* **18**, 185–199. Axisymmetric bending of functionally graded circular and annular plates.
22. E. DIEULESAINT and D. ROYER 1980 *Elastic Waves in Solids*. New York: John Wiley.
23. Y. Y. TANG and K. XU 1994 *International Journal of Engineering Science* **32**, 1579–1591. Exact solutions of piezoelectric materials with moving screw and edge dislocation.
24. A. SILBIGER 1962 *Journal of the Acoustical Society of America* **34**, 862. Non-axisymmetric modes of vibrations of thin spherical shell.